Bound state equation for 4 or more relativistic particles.

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Abstract

We apply the 3D reduction method we recently proposed for the N-particle Bethe-Salpeter equation to the 4-particle case. We find that the writing of the $N \geq 4$ Bethe-Salpeter equation is not a straightforward task, owing to the presence of mutually unconnected interactions, which could lead to an overcounting of some diagrams in the resulting full propagator. We overcome this difficulty in the $N=4$ case by including three counterterms in the Bethe-Salpeter kernel. The application of our 3D reduction method to the resulting Bethe-Salpeter equation suggests us a modified 3D reduction method, which gives directly the 3D potential, without the need of writing the Bethe-Salpeter kernel explicitly. The modified reduction method is usable for all N.

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Contents

1 Introduction.

The Bethe-Salpeter equation [1, 2] is the usual tool for computing relativistic bound states [1-20]. The principal difficulty of this equation comes from the presence of N-1 (for N particles) unphysical degrees of freedom: the relative time-energy degrees of freedom. In a recent work [20], we built a 3D reduction of the two-fermion Bethe-Salpeter equation around an unspecified instantaneous approximation of the Bethe-Salpeter kernel, the difference with the exact kernel being corrected in a series of higher-order contributions to the final 3D potential. This potential being not explicitly hermitian, we performed a second symmetrizing series expansion at the 3D level. The result of the combination of both series gave us an hermitian potential which turned out to be independent of the starting instantaneous approximation, and was in fact a compact expression of the potential that Phillips and Wallace [19] compute order by order. We found that this 3D potential was also obtainable directly by starting with an approximation of the free propagator, based on integrals in the relative energies instead of the more usual $\delta-$constraint (integrating propagator-based reduction). In this form, the method was easily generalizable to a system of N particles, consisting of any combination of bosons and fermions. Special cases of N=2 or 3-fermion systems were examined. Taking the retarded part of the full propagator at equal times, following Logunov and Tavkhelidze [4] (for two particles) and Kvinikhidze and Stoyanov [21] (for three particles) also leads to the same 3D equations.

For more detailed explanations about our integrating propagator-based reduction, about related problems like Lorentz covariance, cluster separability or continuum dissolution, or for comparisons with other approaches, we refer to our preceding work [20].

In the present work we test our reduction method in the four-body problem. The first step is of course the writing of the starting Bethe-Salpeter equation

itself. It is an homogeneous equation for a Bethe-Salpeter amplitude, which is derived by identifying the residues of the bound state poles in the inhomogeneous Bethe-Salpeter equation for the full propagator G. The interactions are introduced via the Bethe-Salpeter kernel K, which is such that the inhomogeneous Bethe-Salpeter equation reproduces, by iterations, the full propagator G as deduced by the Feynman's graphs method. The writing of the Bethe-Salpeter kernel K is straightforward in the two and three-body problems. In the four-body problem, however, we meet a new difficulty due to the mutual unconnectedness of the three pairs of two-body irreducible kernels (12)(34), (13)(24) and (14)(23): adding simply the six two-body irreducible kernels (plus the four irreducible three-body kernels and the irreducible four-body kernel) would lead to an overcounting of some graphs in the expansion of G. We would get for example the sequences of kernels (12)(34) and (34)(12), which are in fact a unique graph or part of graph (unlike sequences like (12)(23) and (23)(12) which are different). It is however not very difficult to correct this overcounting by including three counterterms in K. Computing then the first terms of the 3D potential by our 3D reduction method, we find that, at the often used approximation in which we keep only the two-body kernels, the (relative and total) energy-independent part of these kernels and the positive-energy part of the propagators, the final 3D potential is simply the sum of the six two-particle potentials.

It is clear that the difficulty of the overcounting of the sequences of unconnected kernels will not be so easily overcomed for $N \geq 5$. This suggested us to modify our 3D reduction method in order to avoid the explicit writing of the Bethe-Salpeter kernel K. Our 3D potential is now written directly in a straightforward way for any number of particles, with simply the additional prescription of removing the duplicating diagrams which appear when $N \geq 4$.

In section 2, we recall our 3D reduction method of [20] for the N-body Bethe-Salpeter equation. In section 3 we apply it to the four-body problem. In section 4 we suggest a general 3D reduction method starting directly from the expansion of the full propagator G, as given by Feynman graphs, without the need of writing explicitly the kernel K of the Bethe-Salpeter equation. Section 5 is devoted to conclusions.

2 Integrating propagator-based reduction for the N-body Bethe-Salpeter equation.

The inhomogeneous and homogeneous Bethe-Salpeter equations can be written

$$G = G^0 + G^0 K G \tag{1}$$

$$\Phi = G^0 K \, \Phi \tag{2}$$

where G is the full propagator and Φ the Bethe-Salpeter amplitude. The free propagator G^0 for a system of $f=0,1...N$ fermions and b = N-f bosons is

$$G^0 \,=\, G_1^0 \,...\, G_f^0 \, G_{f+1}^0 \,...\, G_N^0 \,, \tag{3}$$

the propagators of the fermion i and the boson j being respectively

$$G_i^0 \,=\, \frac{1}{p_{i0} - h_i + i\epsilon h_i}\, \beta_i \,, \tag{4}$$

$$G_j^0 \,=\, \frac{1}{p_{j0}^2 - E_j^2 + i\epsilon} \,=\, \frac{1}{2E_j} \sum_{\sigma_j} \frac{\sigma_j}{p_{j0} - \sigma_j E_j + i\epsilon\sigma_j} \,, \tag{5}$$

where p_i is the 4-momentum of particle i , and

$$h_i = \vec{\alpha}_i \,.\vec{p}_i + \beta_i \, m_i \,, \qquad E_i = (\vec{p}_i^{\,2} + m_i^2)^{\frac{1}{2}} \,, \qquad \sigma_j = \pm 1. \tag{6}$$

We do not specify the reference frame in which we write noncovariant quantities like h_i or E_i. Our 3D reduction will in fact be frame-dependent. Practically, we shall choose the global rest frame. The kernel K will be chosen in such a way that equation (1) gives by iterations the usual expansion of G in terms of Feynman graphs. It is related to the kernel K' defined in the same way with the dressed propagators

$$G \,=\, G'^0 \,+\, G'^0 K' G \tag{7}$$

where G'^0 is the product of the dressed propagators

$$G_i'^0 \,=\, \frac{1}{\gamma_i \cdot p_i \,-\, m_i \,-\, \Sigma_i \,+ i\epsilon} \qquad \text{for a fermion,} \tag{8}$$

$$G_i'^0 \,=\, \frac{1}{p_{i0}^2 - E_i^2 - \Sigma_i + i\epsilon} \qquad \text{for a boson,} \tag{9}$$

and Σ_i the renormalized self-energy function. The transfer of the self-energies from the propagator to the kernel is achieved with [20, 22]

$$K \,=\, K' \,+\, \Sigma, \qquad \Sigma \,=\, (G^0)^{-1} - (G'^0)^{-1}. \tag{10}$$

In the two-body problem K' is simply the sum of all irreducible Feynman graphs. In the three-body problem, it is

$$K' \,=\, K'_{12}\,(G_3'^0)^{-1} \,+\, K'_{23}\,(G_1'^0)^{-1} \,+\, K'_{31}\,(G_2'^0)^{-1} \,+\, K'_{123} \,, \tag{11}$$

where K'_{ij} is the sum of the two-body (ij) irreducible Feynman graphs while K'_{123} is the sum of the three-body connected irreducible Feynman graphs. In the four-body problem and beyond, there appear mutually unconnected kernels in the expression of K' (like K'_{12} and K'_{34} for example), and we shall have to be careful to avoid overcountings in the expansion of G (see next section).

In ref. [20], we built a 3D reduction of the two-fermion Bethe-Salpeter equation around an instantaneous approximation of the Bethe-Salpeter kernel, the difference with the exact kernel being corrected in a series of higher-order contributions to the final 3D potential. This potential being not explicitly hermitian, we performed a second symmetrizing series expansion at the 3D level. The result of the combination of both series turned out to be independent of the starting instantaneous approximation, and was also obtainable directly by starting with an approximation of the free propagator, based on integrals in the relative energies instead of the more usual $\delta-$constraints. Furthermore, the method was easily generalizable to a system of N particles, consisting of any combination of bosons and fermions. In the present work, we shall however present our 3D reduction in yet another way. Let us first define the 3D free propagator

$$\int dp_0\, G^0(p_0) \equiv \int \delta\,(P_0 - \sum_{i=1}^{N} p_{i0})\, dp_{10}...dp_{N0}\, G^0(p_{10},...p_{N0}) = \frac{(-2i\pi)^{N-1}}{\omega}\ \tau\, g^0\, \beta \tag{12}$$

with

$$g^0 = \frac{1}{P_0 - S + i\epsilon P_0}, \qquad S = E\,(\Lambda^+ - \Lambda^-)\,, \qquad \beta = \beta_1...\beta_f\,, \tag{13}$$

$$\Lambda^\pm = \Lambda_1^\pm \ldots \Lambda_f^\pm\,, \qquad \Lambda_i^\pm = \frac{E_i \pm h_i}{2E_i}\,, \qquad \tau = \Lambda^+ + (-)^{f+1}\Lambda^-\,, \tag{14}$$

$$E = \sum_{i=1}^{N} E_i\,, \qquad \omega = 2^b\, E_{f+1} \ldots E_N \tag{15}$$

for $f \neq 0$, so that

$$\tau\, g^0 = \frac{\Lambda^+}{P_0 - E + i\epsilon} + (-)^{f+1}\frac{\Lambda^-}{P_0 + E - i\epsilon}. \tag{16}$$

When $f=0$, (bosons only), we have no τ and no β (we can replace them by 1 in (12)) and

$$g^0 = \frac{1}{P_0 - E + i\epsilon} - \frac{1}{P_0 + E - i\epsilon} = \frac{2E}{P_0^2 - E^2 + i\epsilon}\,. \tag{17}$$

Let us now define a 3D full propagator in the τ^2 subspace, with τg^0 as free limit:

$$g = \frac{1}{(-2i\pi)^{N-1}}\,\tau\sqrt{\omega}\int dp'_0 dp_0\, G(p'_0,p_0)\,\beta\,\tau\sqrt{\omega}. \tag{18}$$

The integrations with respect to the relative energies preserve the positions of the bound state poles of G (it can also be shown that the physical particle-particle, particle-bound state or bound state-bound state scattering amplitudes are also preserved [21, 20]). If we write G as

$$G \,=\, G^0 + G^0\, T\, G^0\,, \qquad T \,=\, K(\,1 - G^0 K\,)^{-1} \tag{19}$$

we get

$$g \,=\, \tau g^0 + \tau g^0 < T > \tau g^0 \tag{20}$$

with the definition

$$< A >= \frac{1}{(-2i\pi)^{N-1}}\,\frac{\tau^2\sqrt{\omega}}{g^0}\int dp_0' dp_0\, G^0(p_0')A(p_0',p_0)G^0(p_0)\,\beta\,\frac{\tau^2\sqrt{\omega}}{g^0}. \tag{21}$$

To the 3D full propagator g we can associate the 3D equation

$$\phi \,=\, g^0\,\tau\, V\,\phi \tag{22}$$

where V is such that

$$g^{-1} \,=\, \tau\,(g^0)^{-1} \,-\, V. \tag{23}$$

The comparison with (20) gives

$$< T >= V(1 - \tau g^0 V\,)^{-1} \tag{24}$$

or, conversely

$$V =< T > (1{+}\tau g^0 < T >)^{-1} =< K(1{-}G^0 K)^{-1} > (\,1{+}\tau g^0 < K(1{-}G^0 K)^{-1} >)^{-1}$$

$$= < K(1{-}G^0 K)^{-1}(\,1{+} > \tau g^0 < K(1{-}G^0 K)^{-1})^{-1} >$$

$$= < K(\,1{-}G^0 K{+} > \tau g^0 < K)^{-1} > = < K(\,1{-}G^R K)^{-1} > = < K^T > \tag{25}$$

with

$$K^T \,=\, K(1 - G^R K)^{-1} \,=\, K + K G^R K + \cdots \tag{26}$$

$$G^R \,=\, G^0 \,-\, G^I\,, \qquad G^I \,=\, > \tau g^0 < \tag{27}$$

or, making the dependence on the relative energies explicit:

$$G^R(p_0', p_0) = G^0(p_0)\,\delta\,(p_0' - p_0) - G^I(p_0', p_0) \tag{28}$$

with

$$G^I(p_0', p_0) \,=\, G^0(p_0')\,\beta\,\frac{\tau\,\omega}{(-2i\pi)^{N-1}\, g^0}\, G^0(p_0). \tag{29}$$

By substracting G^I from G^0, we remove the leading terms which come from the residues of the positive-energy poles of G^0. This ensures in principle the decreasing of the terms of the series (26).

The choice (18) of g as the integral of G between two τ operators defines our 3D reduction (in the two-body case it is also Phillips and Wallace's reduction). A different approach, in the two-body case, the constraint approach, consists in defining $< T >$ by fixing the relative-energy arguments of $T(p'_0, p_0)$, for example by putting one initial particle and one final particle on their positive-energy mass shell [16]. There is no problem with this approach for $N=2$. For $N=3$, we should put two initial and two final particles on their positive-energy mass shell. For the two-body terms, this would give a constraint too much. This difficulty can be avoided by applying different constraints to the different terms of T [23, 24]. This leads to a set of coupled equations which can not, without approximations, be reduced to a single equation.

In [20], we derived equation (22) from the inhomogeneous Bethe-Salpeter equation (2), with the definition

$$\phi \,=\, \tau \sqrt{\omega} \int dp_0 \, \Phi(p_0) \tag{30}$$

A simpler reduction method, which we shall adopt from now, can be obtained by keeping only the positive-energy part Λ^+ of τ. In this case, the equation obtained after the reduction of the Dirac spinors into Pauli spinors will be [20]

$$(P_0 - E\,)\,\varphi \,=\, \left[\prod_{i=1}^{f} \sqrt{\frac{2\,E_i}{E_i + m}}\,\right] \; v \; \left[\prod_{i=1}^{f} \sqrt{\frac{2\,E_i}{E_i + m}}\,\right] \; \varphi. \tag{31}$$

where v is the large-large part of V. This simpler reduction method can be defined by the choice of g as the integral of G, now between two positive-energy projectors Λ^+ (instead of τ in (18)). This 3D full propagator is, in configuration space, the retarded part of G taken at equal times. It has been the starting point of a 3D reduction by Logunov and Tavkhelidze [4] (in the two-fermion case), followed by Kvinikhidze and Stoyanov [21] (three fermions) and Khvedelidze and Kvinikhidze [25] (four fermions).

3 The four-body problem.

In this section 3, we shall neglect, for simplicity, the radiative corrections (omitting thus the $\Sigma' s$ and all the primes in the expression of K).

3.1 Bethe-Salpeter equation for four particles.

For four particles, we have [26, 25, 27]

$$G^0 \,=\, G_1^0\, G_2^0\, G_3^0\, G_4^0 \tag{32}$$

$$K \,=\, K_{12,34} \,+\, K_{13,24} \,+\, K_{14,23}$$

$$+\; K_{123}\,(G^0_4\,)^{-1} \;+\; K_{124}\,(G^0_3\,)^{-1} \;+\; K_{134}\,(G^0_2\,)^{-1} \;+\; K_{234}\,(G^0_1\,)^{-1}$$
$$+\;\; K_{1234}\,, \tag{33}$$

with

$$K_{12,34} \;=\; K_{12}\,(G^0_3\,G^0_4\,)^{-1} \;+\; K_{34}\,(G^0_1\,G^0_2\,)^{-1} \;-\; K_{12}\,K_{34} \tag{34}$$

and similarly for $K_{13,24}$ and $K_{14,23}$. The three lines of (33) contain the two, three and four-body kernels respectively. The last term of (34) is a counterterm which has no equivalent in the two and three-body problems. If we write the expansion of G

$$G \;=\; G^0 \;+\; G^0\,K\,G^0 \;+\; G^0\,K\,G^0\,K\,G^0 \;+\cdots \tag{35}$$

and collect the terms containing one K_{12} with one K_{34}, we get indeed

$$G \;=\; \cdots \;-\; G^0K_{12}K_{34}G^0 \;+\; G^0K_{12}(G^0_3\,G^0_4\,)^{-1}G^0K_{34}(G^0_1\,G^0_2\,)^{-1}G^0$$
$$+\; G^0K_{34}(G^0_1\,G^0_2\,)^{-1}G^0K_{12}(G^0_3\,G^0_4\,)^{-1}G^0 \;+\;\cdots \tag{36}$$

Since K_{12} and K_{34} commute mutually, the two last terms of (36) correspond to the same graph which appears thus twice, while the first term is again the same graph with a minus sign. This mechanism works at all orders. The kernel $K_{12,34}$ is indeed given by

$$K_{12,34} \;=\; (G^0\,)^{-1} - (G_{12,34})^{-1} \tag{37}$$

where $G_{12,34}$ is the full propagator obtained by keeping only the interactions K_{12} and K_{34}. It factorizes into

$$(G_{12,34})^{-1} \;=\; (G_{12})^{-1}(G_{34})^{-1} \;=\; \left[\,(G^0_1\,G^0_2\,)^{-1} - K_{12}\,\right]\,\left[\,(G^0_3\,G^0_4\,)^{-1} - K_{34}\,\right] \tag{38}$$

so that (37) gives (34). This problem of commutating kernels was not present in the two and three-body problems. It has been easily overcomed here in the four-body problem. It will become much more complicated in the five-body problem and beyond [26, 27].

3.2 3D reduction of the four-body Bethe-Salpeter equation.

For definiteness we shall work on the Bethe-Salpeter equation for four fermions and perform the reduction based on the positive-energy part of g^0. Then:

$$\Lambda^+\,g^0 \;=\; \frac{\Lambda^+}{P_0 - E + i\epsilon} \tag{39}$$

$$< A > = \frac{1}{(-2i\pi)^3}\,\Lambda^+(P_0 - E\,)\int dp'_0dp_0\,G^0(p'_0)A(p'_0,p_0)G^0(p_0)\,\beta\,\Lambda^+(P_0 - E\,) \tag{40}$$

and the 3D equation will be

$$\phi \,=\, g^0\, V\, \phi \tag{41}$$

with

$$\begin{aligned} V \,&= < K^T > = < K > + < K G^R K > + \cdots \\ &= < K > + < K (G^0 -> g^0 <) K > + \cdots \\ &= < K > + < K G^0 K > - < K > g^0 < K > + \cdots \end{aligned} \tag{42}$$

We shall now compute the two and four-vertex terms, keeping only the two-fermion kernels:

$$\begin{aligned} < K^T >^{(4)} = \quad & < K_{12,34} > + \cdots \quad (3\ terms) \\ + \big[< K_{12,34}\, G^0 K_{14,23} > - & < K_{12,34} > \ g^0 < K_{14,23} > \big] + \cdots \quad (6\ terms) \\ + \big[< K_{12,34}\, G^0 K_{12,34} > - & < K_{12,34} > \ g^0 < K_{12,34} > \big] + \cdots \quad (3\ terms) \end{aligned} \tag{43}$$

where

$$< K_{12,34} > = < K_{12}\, (G_3^0\, G_4^0\,)^{-1} > + < K_{34}\, (G_1^0\, G_2^0\,)^{-1} > - < K_{12}\, K_{34} > \tag{44}$$

$$\begin{aligned} & < K_{12,34}\, G^0 K_{14,23} >^{(4)} = < K_{12}\, (G_3^0\, G_4^0\,)^{-1} G^0 K_{23}\, (G_1^0\, G_4^0\,)^{-1} > \\ & + < K_{12}\, (G_3^0\, G_4^0\,)^{-1} G^0 K_{14}\, (G_2^0\, G_3^0\,)^{-1} > + \ < K_{34}\, (G_1^0\, G_2^0\,)^{-1} G^0 K_{23}\, (G_1^0\, G_4^0\,)^{-1} > \\ & + \ < K_{34}\, (G_1^0\, G_2^0\,)^{-1} G^0 K_{14}\, (G_2^0\, G_3^0\,)^{-1} > \end{aligned} \tag{45}$$

$$\begin{aligned} & < K_{12,34}\, G^0 K_{12,34} >^{(4)} = < K_{12}\, (G_3^0\, G_4^0\,)^{-1} G^0 K_{12}\, (G_3^0\, G_4^0\,)^{-1} > \\ & + \ < K_{34}\, (G_1^0\, G_2^0\,)^{-1} G^0 K_{34}\, (G_1^0\, G_2^0\,)^{-1} > + \ < K_{12}\, (G_3^0\, G_4^0\,)^{-1} G^0 K_{34}\, (G_1^0\, G_2^0\,)^{-1} > \\ & + \ < K_{34}\, (G_1^0\, G_2^0\,)^{-1} G^0 K_{12}\, (G_3^0\, G_4^0\,)^{-1} > \end{aligned} \tag{46}$$

The various terms of (44)-(46) are represented in figure 1. The two last terms of (46) are equal to $< K_{12}\, K_{34} >$. In the expansion of G this term will thus appear only once, with a plus sign.

Let us now examine the counterterms $- < K > g^0 < K >$, which could be represented by the 8 last graphs of figure 1, with a vertical line separating them into two parts. In the two and three-particle problems we have a cancellation between the corresponding contributions of $< K G^0 K >$ and of $- < K > g^0 < K >$, when the two-body kernels are approximated by (relative and total) energy independent kernels and the propagators by positive-energy propagators. This cancellation of the leading terms leads in principle to a decreasing of the contributions of the higher-order terms of the series $< K > + < K G^R K > + \cdots$. We shall check this cancellation here in (43) by assuming that the two-body kernels contain positive-energy projectors and are independent on the energies ($K_{ij} = \Lambda_i^+ \Lambda_j^+ K_{ij} \beta_i \beta_j \Lambda_i^+ \Lambda_j^+$ and is independent of p_{0ij}, P_{0ij} – we shall speak of the "positive-energy instantaneous approximation", although "instantaneous" usually refers only to the independence on the

two-body relative energy). It is then easy to verify that the cancellation occurs for the 4th to the 9th graphs of figure 1. We remain thus with

$$\begin{aligned} < K^T >^{(4)} = \quad & < K_{12}\,(G_3^0\,G_4^0\,)^{-1} > \,+\, < K_{34}\,(G_1^0\,G_2^0\,)^{-1} > \,+ \cdots \\ + < K_{12}\,K_{34} > \,-\, & < K_{12}\,(G_3^0\,G_4^0\,)^{-1} > g^0 < K_{34}\,(G_1^0\,G_2^0\,)^{-1} > \\ - & < K_{34}\,(G_1^0\,G_2^0\,)^{-1} > g^0 < K_{12}\,(G_3^0\,G_4^0\,)^{-1} > \,+ \cdots \end{aligned} \tag{47}$$

The two-vertex contributions become simply the sum of the six two-fermion potentials:

$$V \;=\; V_{12} \,+\, V_{34} \,+\, V_{13} \,+\, V_{24} \,+\, V_{14} \,+\, V_{23}\,, \tag{48}$$

$$V_{ij} \;=\; -2i\pi\,\beta_i\beta_j\,K_{ij}. \tag{49}$$

For $< K_{12}\,K_{34} >$, we have

$$\begin{aligned} < K_{12}\,K_{34} > = & \frac{-1}{2i\pi}\int dP_{120}\,dP_{340}\;\delta\,(P_0 - P_{120} - P_{340}) \\ (P_0 - E') & \left[\frac{1}{P_{120} - E_1' - E_2' + i\epsilon}\;V_{12}\;\frac{1}{P_{120} - E_1 - E_2 + i\epsilon}\right] \\ & \left[\frac{1}{P_{340} - E_3' - E_4' + i\epsilon}\;V_{34}\;\frac{1}{P_{340} - E_3 - E_4 + i\epsilon}\right](P_0 - E). \end{aligned} \tag{50}$$

Let us perform the integration with respect to P_{120} and close the integration path clockwise. We have to consider the poles at $P_{120} = E_1 + E_2$ and $P_{120} = E_1' + E_2'$. We obtain

$$\begin{aligned} < K_{12}\,K_{34} > = & \frac{P_0 - E'}{E_1 + E_2 - E_1' - E_2'}\;V_{12}\;\frac{1}{P_0 - E_1 - E_2 - E_3' - E_4'}\;V_{34} \\ & +\;V_{12}\;\frac{1}{E_1' + E_2' - E_1 - E_2}\;V_{34}\;\frac{P_0 - E}{P_0 - E_1' - E_2' - E_3 - E_4}\,. \end{aligned} \tag{51}$$

Writing then

$$P_0 - E' \;=\; (P_0 - E_1 - E_2 - E_3' - E_4'\,) \,+\, (E_1 + E_2 - E_1' - E_2'\,) \tag{52}$$

$$P_0 - E \;=\; (P_0 - E_1' - E_2' - E_3 - E_4\,) \,+\, (E_1' + E_2' - E_1 - E_2\,) \tag{53}$$

in (51), we see that the contributions of the first terms cancel mutually, so that we remain with

$$< K_{12}K_{34} > = V_{12}V_{34}\left[\frac{1}{P_0 - E_1 - E_2 - E_3' - E_4'} + \frac{1}{P_0 - E_1' - E_2' - E_3 - E_4}\right]. \tag{54}$$

These two terms are cancelled by the two last terms of (47) respectively.

We have thus seen that, in the case of instantaneous positive-energy kernels, the contribution of the four-vertex terms vanishes, while the contribution of the two-vertex terms is simply the sum of the six two-fermion potentials. Let us now examine a typical set of six-vertex graphs, the ones which contain two K_{12} with one K_{34}. Their contributions to the potential are represented in figure 2, with a vertical line when G^0 is to be replaced by $> g^0 <$. After the mutual cancellations of some identical contributions, we obtain the corresponding contribution to $< T >$, minus the four graphs containing one $> g^0 <$, plus the three graphs containing two $> g^0 <$. Let us now consider again the case of instantaneous potentials with positive-energy propagators. We have shown above that, in a sequence $K_{12}\, K_{34}$, we must consider the two possible orders. Here, in a sequence $K_{12}\, K_{12}\, K_{34}$, it can be shown that we must consider the three possible orders, so that the first graph of figure 2, e.g., becomes equal to the sum of the three last ones. In figure 3, we expand the graphs of figure 2 in the case of instantaneous kernels with positive-energy propagators, and we see that the sum of the resulting graphs is zero.

4 Bypassing the Bethe-Salpeter equation.

In section 3, we presented a 3D reduction method for the N-body Bethe-Salpeter equation. The starting homogeneous equation was written in terms of a kernel K, which was designed to reproduce the full propagator G by iterations of the inhomogeneous equation. We have seen that the writing of this kernel K was straightforward for $N = 2, 3,$ less straightforward for $N = 4,$ and increasingly complicated for $N \geq 5$. Our investigations on the $N = 4$ case in section 3 suggests us a possible way of avoiding the explicit writing of the Bethe-Salpeter equation. We saw in section 2 that the 3D potential is given by

$$V \;=< T(1 + G^I T\,)^{-1} >=< K(1 - G^R K\,)^{-1} > \qquad (55)$$

where $G^R = G^0 - G^I,$ while K contain counter-terms in order to avoid the apparition of topologically identical diagrams in the expansion of $T = K(1 - G^0 K)^{-1}$. It is however possible to adopt a simpler algorithm for T :

$$T \;=\; \left[K^{IR}(1 - G^0 K^{IR})^{-1}\right]^{SC} . \qquad (56)$$

By K^{IR} we denote the sum of the irreducible interactions (as (33-34) without the three counter-terms) and by SC (single counting), we mean that the diagrams which appear two or more times in the expansion of T must be kept only once. The 3D potential will then be written

$$V \;=< K^{IR}(1 - G^R K^{IR})^{-1} >^{SC} \qquad (57)$$

in which, after the expansion of the series and the splitting of G^R into $G^0 - G^I$, we shall remove the duplicating diagrams. This makes the writing of the 3D

potential as straightforward for $N \geq 4$ as it was for $N = 1$ or 2. As an example in the four-particle case:

$$\begin{aligned} & (K_{12} G^R K_{34} \,+\, K_{34} G^R K_{12})^{SC} \\ &= (K_{12} G^0 K_{34} \,+\, K_{34} G^0 K_{12} \,-\, K_{12} G^I K_{34} \,-\, K_{34} G^I K_{12})^{SC} \\ &= K_{12} G^0 K_{34} \,-\, K_{12} G^I K_{34} \,-\, K_{34} G^I K_{12}. \end{aligned} \tag{58}$$

With positive-energy instantaneous interactions, the term in G^0 will be cancelled by the two terms in G^I.

The writing of the Bethe-Salpeter kernel could be done in a quite similar way:

$$K \,=\, \left\{ T(1 + G^I T)^{-1} \right\}_{G^I \to G^0} \,=\, \left\{ \left[K^{IR} (1 - [\, G^0 - G^I] \, K^{IR})^{-1} \right]^{SC} \right\}_{G^I \to G^0}. \tag{59}$$

Here, G^I is simply a temporary renaming of G^0, which indicates that terms like $AG^I B$ and $BG^I A$ must always be kept both. We know that K, unlike V, contain only a finite number of parts [27]. For $N = 4$, e.g., eqs. (33-34) show that K contains 11 subkernels (groups of irreducible graphs) in K^{IR} plus 3 counterterms, coming from the second order in K^{IR} (we get one of these counterterms by replacing G^I by G^0 in (58)).

The operator T is directly given by the Feynman graphs if we neglect the radiative corrections. If not, we can use

$$G^0 + G^0 T G^0 \,=\, G \,=\, G'^0 + G'^0 T' G'^0 \tag{60}$$

with

$$G'^0 \,=\, G^0 \, (1 - \Sigma \, G^0 \,)^{-1} \,=\, \prod_i G_i^0 \, (1 - \Sigma_i \, G_i^0 \,)^{-1}. \tag{61}$$

The operators G and T' are directly given by the Feynman graphs. For T, we have

$$T \,=\, \Sigma (1 - G^0 \Sigma)^{-1} \,+\, (1 - \Sigma G^0)^{-1} \, T' \, (1 - G^0 \Sigma)^{-1}. \tag{62}$$

5 Conclusions

In a previous work [20], we were in search of a 3D reduction method for the two-fermion Bethe-Salpeter equation based on an unspecified positive-energy instantaneous approximation of the Bethe-Salpeter kernel. We performed a series expansion around this approximation, followed by an integration on the relative energy and a second series expansion, at the 3D level, in order to render the resulting 3D potential symmetric. After combining both series, we found that we had in fact built a kind of propagator-based reduction, using an integration with respect to the relative energy, in full contrast with the usual constraining

propagator-based reductions, which use constraints. Furthermore, this method was easily generalisable to systems consisting in any number of fermions and/or bosons.

In the present work, the increasing difficulty of writing the Bethe-Salpeter kernel for $N \geq 4$ suggested us a direct way of writing the 3D potential without the need of first writing the Bethe-Salpeter kernel explicitly. This writing of the 3D potential is straightforward and valid for all $N,$ with the simple prescription of removing the duplicating graphs which appear when $N \geq 4.$,

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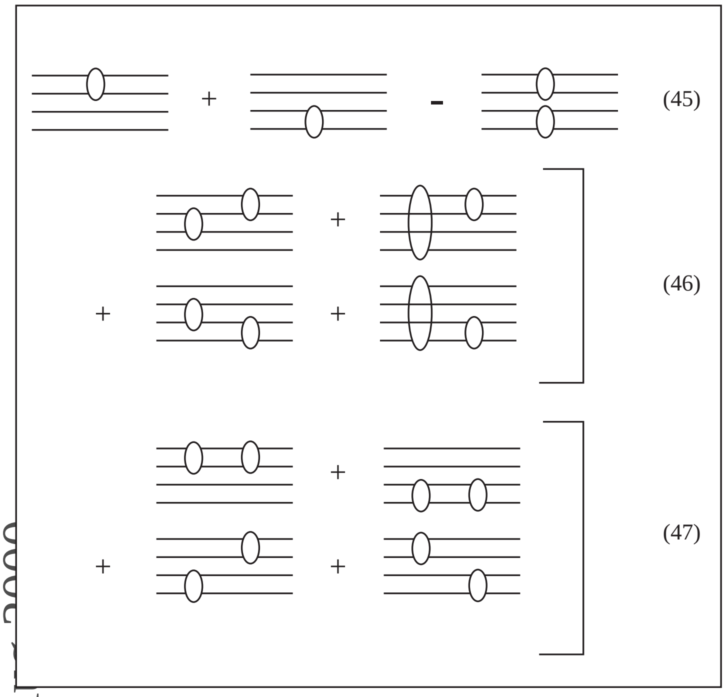

Figure 1. Typical terms of <K+KG^{0}K>.

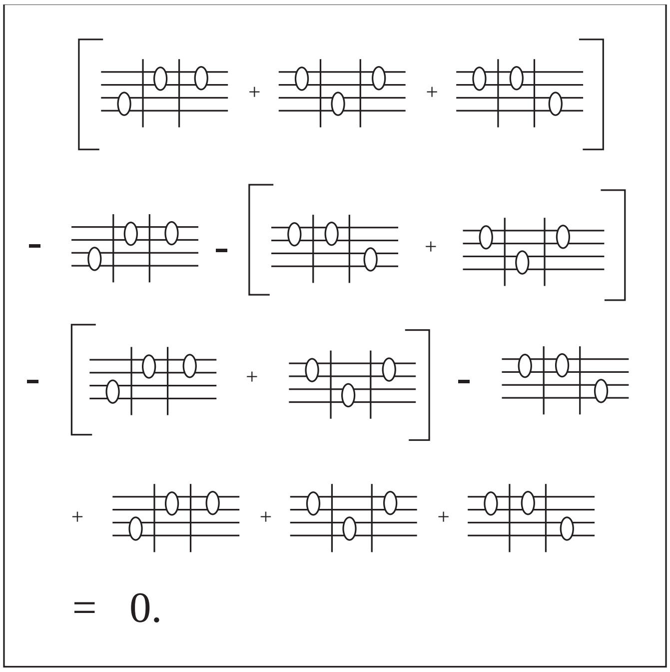

Figure 3. Two (12)-one (34) terms of <KG^{R}KG^{R}K> in the positive-energy instantaneous approximation.

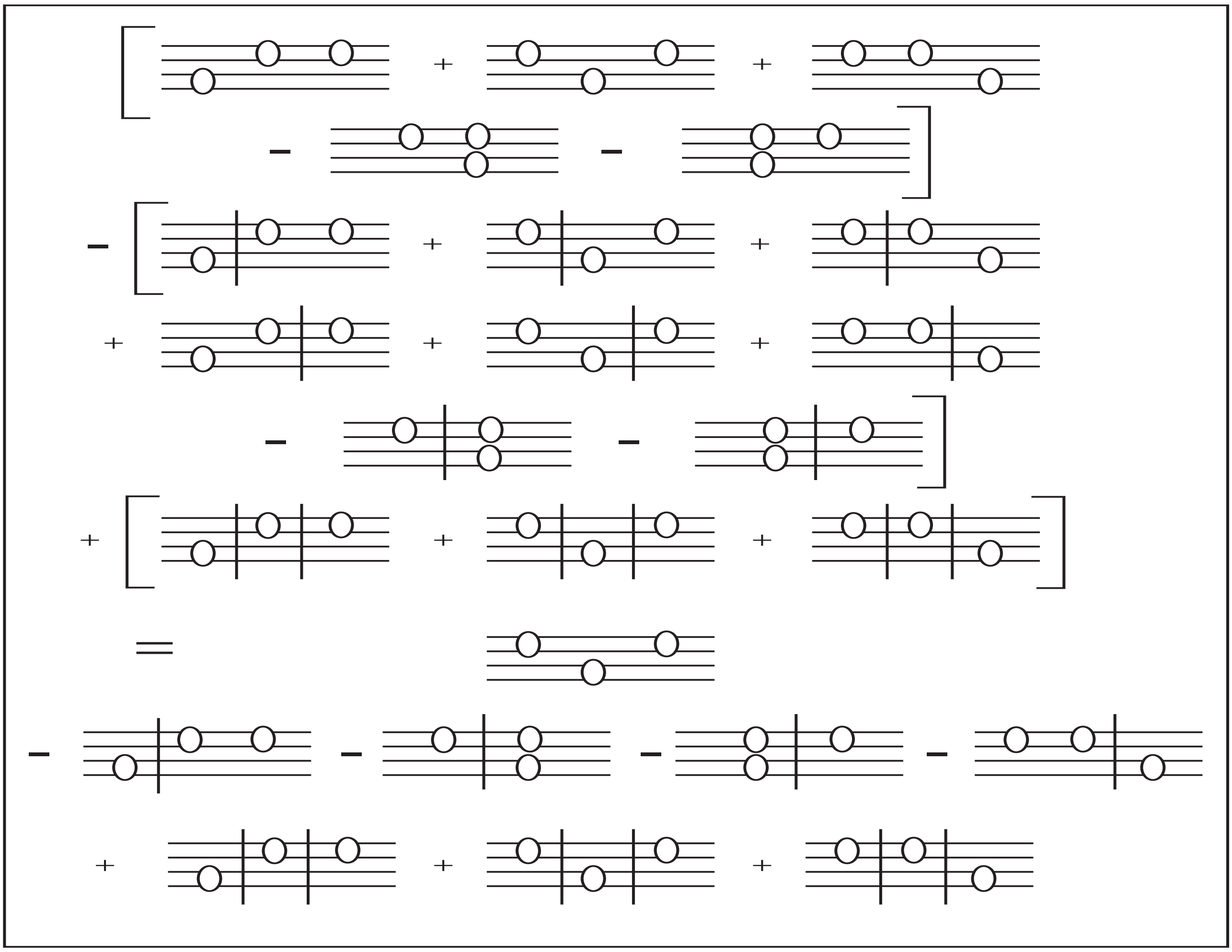

Figure 2. Two (12)-one (34) terms of <KG^{R}KG^{R}K>.